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# A Delta-Kicked Brownian Rotor

**L. E. Reichl**[1]

We study the evolution of a delta-kicked Fokker–Planck equation which is a certain limiting case of a driven Brownian rotor with large friction. The time evolution of the rotor is given by a Floquet map in which the effects of diffusion and of the kick are decoupled. For the case where absorbing boundaries are introduced, we show the mechanism causing an abrupt drop in the average survival time as the kick strength is increased.

**KEY WORDS:** Delta-kicked; Brownian rotor; Fokker–Planck equation.

## 1. INTRODUCTION

Recently, Reichl *et al.*[1,2] have shown that some of the manifestations of chaos that occur in quantum systems can also be seen in the Fokker–Planck equation for a heavily damped, softly driven Brownian rotor. The Fokker–Planck equation considered in refs. 1 and 2 was of the type

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \theta}\left[\lambda \sin(\theta+\alpha)\sin(\omega t)\, P\right] + D_0 \frac{\partial^2 P}{\partial \theta^2} \tag{1.1}$$

where $\theta$ is the angular position of the rotor, $P = P(\theta, t)\, d\theta$ is the probability to find the rotor in the interval $\theta \to \theta + d\theta$ at time $t$, $\lambda$ is the strength of the driving force, $\alpha$ is a constant phase factor, and $D_0$ is the diffusion coefficient. The authors observed resonances between Floquet decay rates, level repulsion in the Floquet spectrum, and a fairly abrupt drop in a certain mean first passage time for the problem. Chen[3] subsequently analyzed the Langevin equation for this system and found that in the regime where the manifestations of chaos occur, the Langevin dynamics undergoes a transition to a type of noise-induced deterministic chaos.

[1] Center for Statistical Mechanics and Complex Systems, University of Texas at Austin, Austin, Texas 78712.



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Much of the work in refs. 1–3 was by necessity numerical since Eq. (1.1) is nonintegrable. In this paper, we wish to consider a somewhat pathological variation of Eq. (1.1). We will replace the soft time dependence $\sin(\omega t)$ by a periodic delta kick $\delta_T(t)$, where

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} + \frac{2}{T} \sum_{m=1}^{\infty} \cos\left(\frac{2\pi m t}{T}\right) \tag{1.2}$$

and we choose $\alpha = 0$. Then the Fokker–Planck equation reads

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \theta} [\lambda \sin(\theta)\, \delta_{T_0} P] + D_0 \frac{\partial^2 P}{\partial \theta^2} \tag{1.3}$$

We can simplify Eq. (1.3) if we let $t = D_0$ and $T = T_0 D_0$. Then the Fokker–Planck equation takes the form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \theta} [\lambda \sin(\theta)\, \delta_T P] + \frac{\partial^2 P}{\partial \theta^2} \tag{1.4}$$

It important to note that while Eq. (1.1) rigorously describes the large-friction behavior of a Brownian rotor, Eqs. (1.3) and (1.4) do not. The method for obtaining the large-friction behavior of a Brownian particle was first described by Kramers[4] and an alternative method was given by Brinkman.[5] In both cases, one expands the Fokker–Planck equation for a particle with moderate friction in powers of the inverse friction and retains only leading terms in the inverse friction. If a time-dependent driving term is present, the expansion will involve the ratio of the frequency of the driving term to the friction. If we would truncate Eq. (1.2) to a finite number of cosine terms, then the truncated versions of Eqs. (1.3) and (1.4) would be valid equations to describe the Brownian rotor. In this paper, we consider the singular limit and keep the delta function. It will give an interesting picture of the contribution of each part of the Fokker–Planck equation and will shed some light on what is happening in the "soft" case.

Delta-kicked systems have proven to be extremely useful in classical and quantum mechanics because they provide a means of strobing the system. In classical mechanics, the dynamical evolution of the delta-kicked rotor can be written in terms of a simple map, called the standard map.[6,7] The standard map maps the phase space of the rotor from one period of the field to the next and therefore gives a stroboscopic picture of the dynamical evolution of the kicked rotor. In quantum mechanics, the time evolution of the Schrödinger equation for the delta-kicked rotor can be written in terms of a Floquet transition matrix which allows a simple mapping of the state of the quantum system from one period to the next.[7,8] We will see that a similar mapping can be obtained for the time evolution of the Fokker–Planck equation (1.4).

As we shall see, analysis of Eq. (1.4) will give us some insight into the behavior of the softly driven rotors because for the delta-kicked case the diffusion process and the kick act independently of one another. We begin in Section 2 by discussing the spectral properties of the kick operator and the diffusion operator. In Section 3 we describe how the Brownian rotor relaxes to equilibrium, and in Section 4 we compute the survival time for the case when absorbing boundaries are applied to the system. Finally, in Section 5 we make some concluding remarks.

## 2. SPECTRAL PROPERTIES

It is possible to write the time evolution of the system described by Eq. (1.4) in terms of a transition matrix which gives the probability distribution at discrete times $T$, where $T$ is the period of the kick. It is convenient to write this evolution matrix in terms of diffusion eigenstates and eigenstates of the kick.

### 2.1. Eigenstates of the Kick

If we integrate across the kick at time $t=T$, Eq. (1.4) takes the form

$$\int_{T^-}^{T^+} dt\,\frac{\partial P}{\partial t} = -\int_{T^-}^{T^+} dt\,\frac{\partial}{\partial \theta}\left[\lambda \sin(\theta)\,\delta_T P\right] + \int_{T^-}^{T^+} dt\,\frac{\partial^2 P}{\partial \theta^2} \tag{2.1}$$

where $T^{\pm} = T \pm \varepsilon$ and $\varepsilon$ is set to zero after the integration. However, since $\partial P/\partial t \approx \delta_T(t)$, the probability $P(\theta, t)$ will have a discrete jump at the kicks, and $\int dt\, P(\theta, t)$ will be continuous, but will have a discontinuous slope at the kicks. Therefore, in the limit $\varepsilon \to 0$, the rightmost term in Eq. (2.1) will not contribute. Thus, the motion across the kick is determined by the equation

$$\frac{\partial P(\theta, t)}{\partial t} = -\frac{\partial}{\partial \theta}\left[\lambda \sin(\theta)\,\delta_T(t)\,P(\theta, t)\right] = -\delta_T(t)\,\lambda \hat{L}_K P(\theta, t) \tag{2.2}$$

where the "kick" operator $\hat{L}_K$ is defined by

$$\hat{L}_K = \cos(\theta) + \sin(\theta)\,\frac{\partial}{\partial \theta} \tag{2.3}$$

Let us now integrate from just before the kick, $t=T^-$, to just after the kick, $t=T^+$. Then

$$P(\theta, T^+)=[\exp(-\lambda\hat{L}_K)]\, P(\theta, T^-) \tag{2.4}$$

and $\exp(-\lambda\hat{L}_K)$ is the evolution operator at the kick.

The kick operator $\hat{L}_K$ is not self-adjoint and therefore has different left and right eigenstates. We will let $\Psi^R_\mu(\theta)$ denote the right eigenstates. The right eigenstates satisfy the equation

$$\hat{L}_K \Psi^R_\mu(\theta)=i\mu\Psi^R_\mu(\theta) \tag{2.5}$$

The differential equation (2.5) has singularities at $\theta=0$ and $\theta=\pi$. However, we can define eigenstates on the interval $0\leqslant\theta\leqslant\pi$. The right eigenstate is

$$\Psi^R_\mu(\theta)=\langle\theta\,|\,\Psi^R_\mu\rangle=\left(\frac{1}{2\pi}\right)^{1/2}\frac{\exp\{i\mu\ln[\tan(\theta/2)]\}}{\sin(\theta)} \tag{2.6}$$

The eigenvalues $\mu$ can take on a continuum of values, and the spectrum of $\hat{L}_K$ is continuous.

The left eigenvectors are eigenvectors of the adjoint operator $\hat{L}^T_K=\sin(\theta)\,\partial/\partial\theta$. The left eigenvectors $\Psi^L_\mu(\theta)$ satisfy the eigenvalue equation

$$\hat{L}^T_K\Psi^L_\mu(\theta)=i\mu\Psi^L_\mu(\theta)$$

and are given by

$$\Psi^L_\mu(\theta)=\langle\theta\,|\,\Psi^L_\mu\rangle=\left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{i\mu\ln\left[\tan\left(\frac{\theta}{2}\right)\right]\right\} \tag{2.7}$$

These eigenstates are complete and orthonormal. That is, they satisfy the completeness condition

$$\int_{-\infty}^{\infty} d\mu\, \psi^{L*}_\mu(\theta')\,\psi^R_\mu(\theta)=\delta(\theta-\theta') \tag{2.8}$$

and orthonormality condition

$$\int_0^\pi d\theta\, \psi^{L*}_\mu(\theta)\,\psi^R_{\mu'}(\theta)=\delta(\mu-\mu') \tag{2.9}$$

where $\delta(\theta-\theta')$ and $\delta(\mu-\mu')$ are Dirac delta functions (see the Appendix).

In Fig. 1 we show a plot of $\mathrm{Re}[\Psi^R_\mu(\theta)]$ for $\mu=9$. There are an infinite number of oscillations in the neighborhood of $\theta=0$ and $\theta=\pi$.

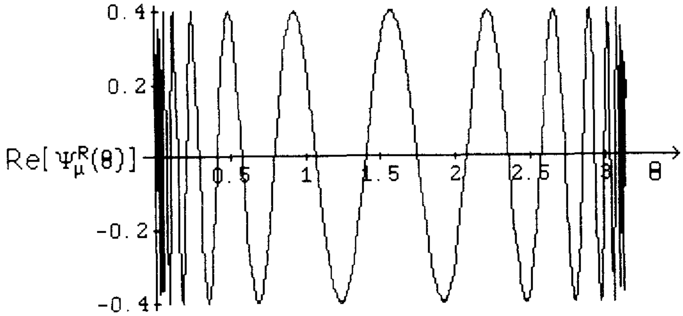


Fig. 1. A plot of Re[$\Psi_\mu^R(\theta)$] versus $\theta$ for $\mu = 9$.

## 2.2. Eigenstates of the Diffusion Operator

We will consider Brownian motion on the interval $0 \to \pi$. Between kicks the rotor evolves according to the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial \theta^2} = \hat{L}_D P \tag{2.10}$$

The diffusion operator $\hat{L}_D = \partial^2/\partial\theta^2$ is a self-adjoint operator. On the interval $0 \leqslant \theta \leqslant \pi$, $\hat{L}_D$ has a discrete spectrum. If we consider periodic boundary conditions on the interval $0 \leqslant \theta \leqslant \pi$, the eigenstates of $\hat{L}_D$ are given by

$$\Phi_n(\theta) = \left(\frac{1}{\pi}\right)^{1/2} \exp(i2n\theta)$$

where $n$ is an integer with range $-\infty \leqslant n \leqslant \infty$. Thus

$$\hat{L}_D \Phi_n(\theta) = -4n^2 \Phi_n(\theta) \tag{2.11}$$

and the eigenvalues of the diffusion operator $\hat{L}_D$ for periodic boundary conditions on the interval $0 \leqslant \theta \leqslant \pi$ are given by $-4n^2$.

## 3. RELAXATION TO EQUILIBRIUM

The probability just before the $N$th kick is related to the probability just before the $(N-1)$th kick by

$$|P(NT^-)\rangle = [\exp(T\hat{L}_D)\exp(-\lambda\hat{L}_K)]\,|P((N-1)\,T^-)\rangle \tag{3.1}$$

and it is related to the initial state by

$$|P(NT^-)\rangle = [\exp(T\hat{L}_D)\exp(-\lambda\hat{L}_K)]^N\,|P(0^-)\rangle \tag{3.2}$$

In order to follow the time evolution of the Brownian particle on the interval $0 \to \pi$, we can evaluate the Floquet transition operator

$$\hat{V}(\lambda, T) = \exp(T\hat{L}_D)\exp(-\lambda\hat{L}_K) \tag{3.3}$$

with respect to eigenstates of the diffusion operator. We then obtain the Floquet transition matrix

$$\begin{aligned} V_{n',n} &= \langle n'|\exp(T\hat{L}_D)\exp(-\lambda\hat{L}_K\,|n\rangle \\ &= \exp(-4n^2T)\int_{-\infty}^{\infty} d\mu\,\langle n'|\psi_\mu^R\rangle\,[\exp(-i\lambda\mu)]\,\langle\psi_\mu^L|n\rangle \end{aligned} \tag{3.4}$$

We need expressions for the states $\langle n|\psi_\mu^R\rangle$ and $\langle\psi_\mu^L|n\rangle$. Let us first consider $\langle n|\psi_\mu^R\rangle$. We can write

$$\begin{aligned} \langle n|\psi_\mu^R\rangle &= \int_0^\pi d\theta\,\langle n|\theta\rangle\langle\theta|\psi_\mu^R\rangle \\ &= \frac{1}{\sqrt{2}\,\pi}\int_0^\pi \frac{d\theta}{\sin(\theta)}\exp(-2in\theta)\exp\left[i\mu\ln\left(\tan\frac{\theta}{2}\right)\right] \end{aligned} \tag{3.5}$$

If we make the change of variables $\theta = \phi + \pi/2$, then Eq. (3.5) takes the form

$$\begin{aligned} \langle n|\psi_\mu^R\rangle = \frac{1}{\sqrt{2}\,\pi}(-1)^n\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\cos(\phi)}\exp(-2in\phi) \\ \times\exp\left[i\mu\ln\frac{1+\tan(\phi/2)}{1-\tan(\phi/2)}\right] \end{aligned} \tag{3.6}$$

Let us now make a last change of variables. We let

$$x = \ln\frac{1+\tan(\phi/2)}{1-\tan(\phi/2)} \tag{3.7}$$

Then $dx = d\phi/\cos(\phi)$, $\sin(\phi) = \tanh(x)$, and $\cos(\phi) = \mathrm{sech}(x)$, and the range of integration extends over the interval $-\infty \leqslant x \leqslant \infty$. (In Fig. 2 we plot $x$ versus $\theta$.) Equation (3.6) then takes the form

$$\langle n|\psi_\mu^R\rangle = \frac{1}{\sqrt{2}\,\pi}(-1)^n\int_{-\infty}^{\infty} dx\,e^{i\mu x}e^{-2in\phi(x)} \tag{3.8}$$

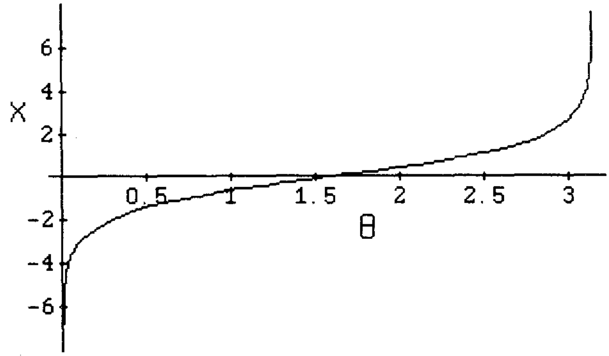


Fig. 2. A plot of $x$ versus $\theta$ for the interval $0.001 \leqslant \theta \leqslant \pi - 0.001$.

Similarly,

$$\langle \psi_\mu^L | n \rangle = \frac{1}{\sqrt{2}\,\pi} (-1)^n \int_{-\infty}^{\infty} dx\, \operatorname{sech}(x)\, e^{-i\mu x} e^{2in\phi(x)} \tag{3.9}$$

If we make use of the definition of the Dirac delta function,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu\, e^{-i\mu(x-x')} = \delta(x - x')$$

the Floquet transition matrix becomes

$$\begin{aligned} V_{n',n} &= \langle n' |\exp(T\hat{L}_D) \exp(-\lambda \hat{L}_K | n \rangle \\ &= \frac{1}{\pi} (-1)^{n-n'} \exp(-4n^2 T) \int_{-\infty}^{\infty} dx\, \operatorname{sech}(x) \\ &\quad \times \exp[-2in'\phi(x+\lambda)] \exp[2in\phi(x)] \end{aligned} \tag{3.10}$$

This Floquet transition matrix can be used to study the distribution of probability after $N$ kicks.

As an example, consider the case when the Brownian particle is located at $\theta = \pi/2$ at time $t = 0^-$. Then, $\langle \theta | P(0^-) \rangle = \delta(\theta - \pi/2)$ and $\langle n | P(0^-) \rangle = \pi^{-1/2} e^{-in\pi}$. After $N$ kicks the probability is

$$\langle \theta | P(NT^-) \rangle = \frac{1}{\pi} \sum_{n'=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2in'\theta} \langle n' | \hat{V}^N | n \rangle e^{-in\pi} \tag{3.11}$$

After a long time ($N \to \infty$), $\langle \theta | P(NT^-) \rangle \to 1/\pi$.

It is interesting to find the spread of probability after one kick. The kick operator is basically a translation operator. At each kick it moves the particle a finite distance along the $\theta$ axis. However, because of the angle dependence of the kick operator, the Brownian particle can never be kicked past the points $\theta = 0$ or $\theta = \pi$. We show this in Fig. 3, where we start the particle at $\theta = \pi/2$ at time $t = 0^-$. We then kick it at time $t = 0$ and look at it again at time $t = T^-$. We have used 30 modes to represent the delta

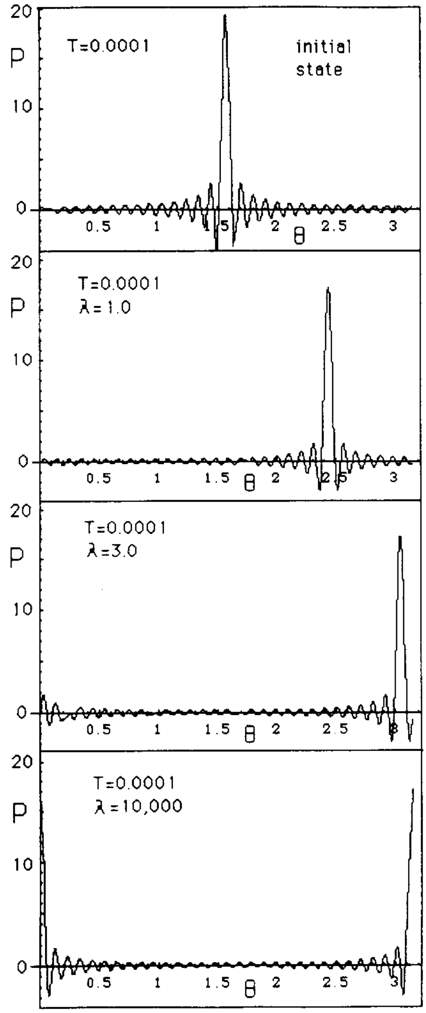


Fig. 3. The upper box shows the initial state of the particle (at time $t = 0^-$) using 30 diffusion eigenstates. The lower three boxes show the position of the particle at time $t = T^-$ for (in descending order) kick strengths $\lambda = 1, 3$, and 10,000. The period between kicks was $T = 0.0001$. (The regions of negative probability occur because we have truncated the Fourier series after 30 terms. The regions of negative probability go to zero as the number of terms in the Fourier series goes to infinity.)

function and we have chosen $T=0.0001$ so that these modes do not decay significantly during one period. Figure 3 shows the position of the Brownian particle after one kick for kick strengths $\lambda=1$, 3, and 10000. The center of the particle never gets out of the region $0 \leqslant \theta \leqslant \pi$ (because of the periodic boundary conditions, the rightmost part of the particle reappears at the left).

## 4. SURVIVAL TIMES

We can study survival times by considering slightly different boundary conditions. Let us put absorbing boundaries at $\theta=0$ and $\theta=\pi$. Then, $\langle\theta \mid P(0^-)\rangle=0$ when $\theta=0$ and $\theta=\pi$. The eigenstates of the diffusion operator $\hat{L}_D$ on the interval $0 \leqslant \theta \leqslant \pi$ are given by

$$\phi_k(\theta)=\left(\frac{2}{\pi}\right)^{1/2} \sin(k\theta) \tag{4.1}$$

where $k$ is an integer with range $1 \leqslant k \leqslant \infty$. The eigenvectors satisfy the eigenvalue equation

$$\hat{L}_D \phi_k(\theta)=-k^2 \phi_k(\theta) \tag{4.2}$$

### 4.1. Transition Operators

In order to study the relaxation of the Brownian particle on the interval $0 \rightarrow \pi$, we can evaluate the Floquet transition operator

$$\hat{V}(\lambda, T)=\exp(T\hat{L}_D) \exp(-\lambda \hat{L}_K) \tag{4.3}$$

with respect to eigenstates $|k\rangle$ of the diffusion operator. We write

$$\begin{aligned} V_{k',k} &= \langle k'|\exp(T\hat{L}_D) \exp(-\lambda \hat{L}_K)|k\rangle \\ &= \exp(-k^2 T) \int_{-\infty}^{\infty} d\mu\, \langle k' \mid \psi_\mu^R\rangle[\exp(-i\lambda\mu)]\langle\psi_\mu^L \mid k\rangle \end{aligned} \tag{4.4}$$

We now need expressions for the states $\langle k \mid \psi_\mu^R\rangle$ and $\langle\psi_\mu^L \mid k\rangle$. Let us first consider $\langle k \mid \psi_\mu^R\rangle$. We can write

$$\begin{aligned} \langle k \mid \psi_\mu^R\rangle &= \int_0^\pi d\theta\, \langle k \mid \theta\rangle\langle\theta \mid \psi_\mu^R\rangle \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \sin\left\{k\left[\phi(x)+\frac{\pi}{2}\right]\right\} e^{i\mu x} \end{aligned} \tag{4.5}$$

Similarly,

$$\langle \psi_\mu^L | k \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \operatorname{sech}(x) \sin \left\{ k \left[ \phi(x) + \frac{\pi}{2} \right] \right\} e^{-i\mu x} \tag{4.6}$$

The Floquet transition matrix becomes

$$\begin{aligned} V_{k',k} &= \langle k' | \exp(T\hat{L}_D) \exp(-\lambda \hat{L}_K) | k \rangle \\ &= \frac{2}{\pi} \exp(-k^2 T) \int_{-\infty}^{\infty} dx \operatorname{sech}(x) \\ &\quad \times \sin \left\{ k' \left[ \phi(x+\lambda) + \frac{\pi}{2} \right] \right\} \sin \left\{ k \left[ \phi(x) + \frac{\pi}{2} \right] \right\} \end{aligned} \tag{4.7}$$

This transition matrix can be used to study the distribution of probability after $N$ kicks.

In Fig. 4 we show the topology of the kick transition matrix $U_{k',k} = \langle k' \exp(-\lambda \hat{L}_K | k \rangle$ for $\lambda = 0.1$, 1, and 2. For small kick strengths only a few neighboring modes are coupled through the kick. However, for larger kick strength many neighboring modes in the Floquet transition matrix are coupled.

In Fig. 5 we show the Floquet transition matrix for kick strength $\lambda = 1.0$ and period $T = 0.01$. Comparison between Figs. 4 and 5 shows clearly the damping of higher modes due to diffusion.

## 4.2. Average Survival Times

Let us now assume that the rotor lies at $\theta = \pi/2$ at time $t = 0^-$. This corresponds to an initial condition $P(\theta, 0^-) = \delta(\theta - \pi/2)$ [$P(\theta, 0^-)$ is actually a conditional probability]. We write

$$P(\theta, 0^-) = \sum_{k=1}^{\infty} b_k(0^-) \, \phi_k(\theta) = \delta(\theta - \pi/2) \tag{4.8}$$

It is easy to show that

$$b_k(0^-) = \left( \frac{2}{\pi} \right)^{1/2} \sin \left( \frac{k\pi}{2} \right) \tag{4.9}$$

In Fig. 6 we show the time evolution of the rotor, given that it is initially localized at $\theta = \pi/2$. The upper box shows the initial state using 40 diffusion eigenstates. The middle box shows the state of the rotor for kick strength $\lambda = 0.1$ after $N = 1$, 10, and 20 kicks. We have used $T = 0.001$ to keep the damping of lower modes small. In the bottom box, we show the state of the rotor for kick strength $\lambda = 1.0$ after $N = 1$ and 2 kicks. In the middle box, the rotor is more spread out than in the bottom box because the higher modes have had a chance to decay away.

The probability that the rotor has "survived," that is, has not hit the absorbing boundaries before time $t = NT^{-}$, is given by

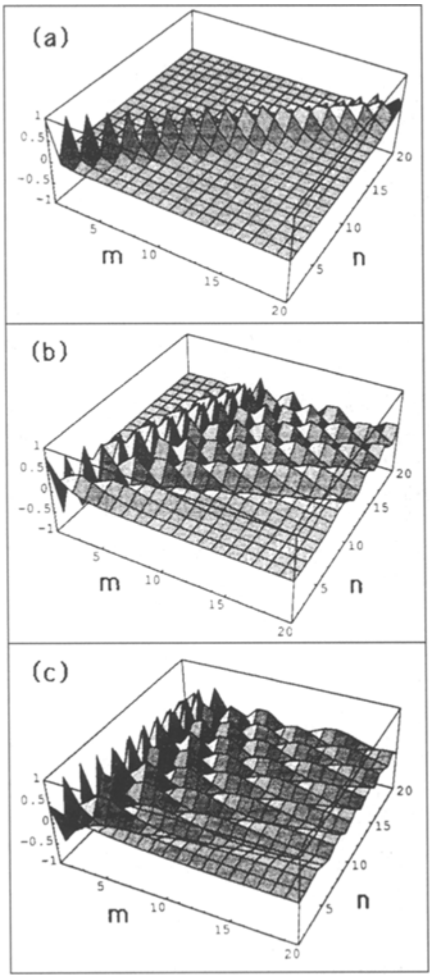


Fig. 4. Topology of the kick transition matrix $U_{k',k}$ (for $k' = m$ and $k = n$) for (a) $\lambda = 0.1$, (b) $\lambda = 1.0$, (c) $\lambda = 2.0$.

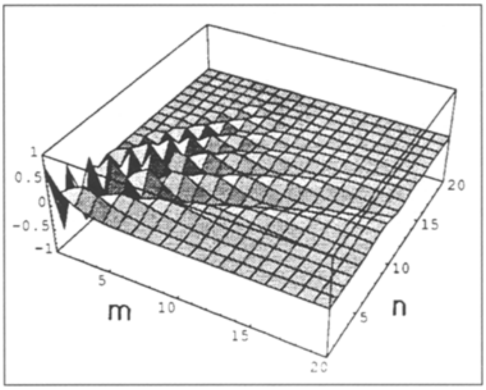


Fig. 5. Topology of the Floquet transition matrix for $\lambda = 1.0$ and $T = 0.01$.

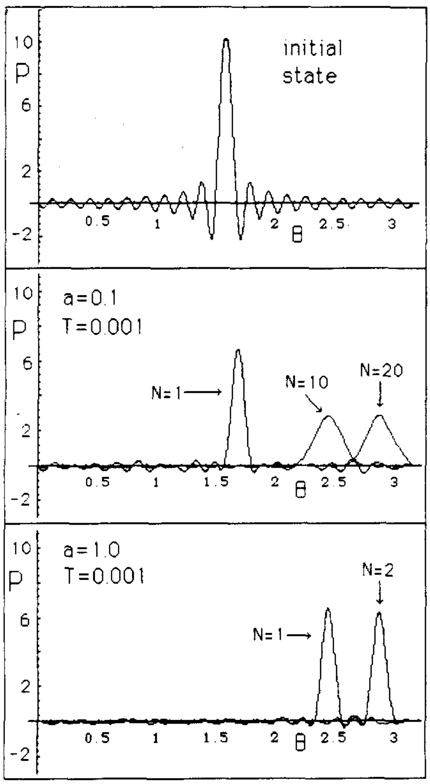


Fig. 6. Time evolution of the rotor for $T = 0.001$, given that it is initially localized at $\theta = \pi/2$. The upper box shows the initial state using 40 diffusion eigenstates. The middle box shows the state of the rotor for kick strength $\lambda = 0.1$ after $N = 1$, 10, and 20 kicks. The bottom box shows the state of the rotor for kick strength $\lambda = 1.0$ after $N = 1$ and 2 kicks. (The regions of negative probability occur because we have truncated the Fourier series after 40 terms. These regions go to zero as the number of terms in the Fourier series goes to infinity.)

$$P(NT^-) = \int_0^\pi d\theta\, P(\theta, NT^-)$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \sum_{k=1(\text{odd})}^{\infty} \frac{2}{k} b_k(NT^-)$$
$$= \frac{2}{\pi} \sum_{k=1(\text{odd})}^{\infty} \sum_{k'=1(\text{odd})}^{\infty} \frac{2}{k} [V^N(T)]_{k,k'} \sin\left(\frac{k'\pi}{2}\right) \tag{4.10}$$

It is possible to compute the average survival time $\langle t \rangle$. It is defined by

$$\langle t \rangle = \frac{\sum_{N=1}^{\infty} NP(NT^-)}{\sum_{N=1}^{\infty} P(NT^-)} \tag{4.11}$$

In Fig. 7 we plot $\langle t \rangle$ as a function of $\lambda$ for different values of the period $T$. The survival time suddenly drops to very low values as we increase the kick strength $\lambda$. When the kick strength is large enough to kick the rotor into the neighborhood of the singularity (where the absorbing boundary is located) higher eigenstates dominate and decay very fast. In Fig. 8 we plot the survival probability as a function of the number of kicks for $\lambda = 0.1$ and several different periods $T$. The results of Figs. 7 and 8 were obtained using a finite-size Floquet transition matrix, but are good because of the rapid decay of the matrix for large values of $n$ (see Fig. 5).

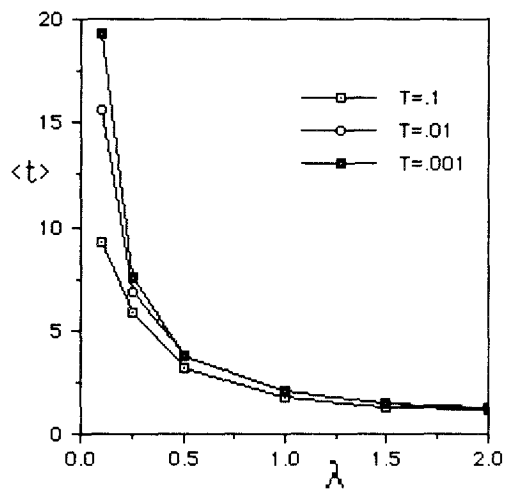


Fig. 7. The average survival time $\langle t \rangle$ versus kick strength $\lambda$.

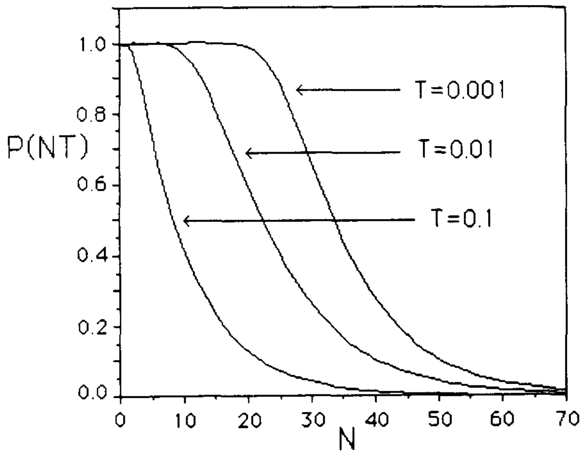


Fig. 8. A plot of the survival probability $P(NT^-)$ for three different periods $T$ and kick strength $\lambda = 0.1$.

## 5. CONCLUSION

The Brownian motion described by Eq. (1.4) is unusual because the influence of the diffusion process and the driving force are completely independent of one another. We have found that the kick operator has singularities at points $\theta = 0$ and $\pi$, and surprisingly it has a continuous spectrum. The eigenstates of the kick operator have an infinite number of oscillations in the neighborhood of the singularities, and the Brownian rotor can never move past the singularities. If we analyze the behavior of the Brownian rotor in terms of eigenstates of the diffusion operator (which has a discrete spectrum), then as the Brownian rotor moves closer to the singularities, the very short-wavelength diffusion eigenstates play an increasingly important role.

The behavior of the delta-kicked rotor gives us some insight into the results of refs. 1–3, where the softly kicked rotor was discussed. In softly kicked Brownian motion, diffusion acts continuously while the force acts. Therefore, the Fokker–Planck equation does not have singularities. However, some of the behavior seen here appears to remain. There is a fast drop in the first passage time when the rotor reaches the "singular" region, and the Floquet spectral statistics undergoes level repulsion. The level repulsion very likely results from the greater connectedness and growth of importance of higher diffusion eigenstates in the singular regions. Certainly, the behavior observed in refs. 1–3 is consistent with what has been observed here.

## APPENDIX

### A.1. Orthonormality

We will show that $\psi_\mu^R(\theta)$ and $\psi_\mu^L(\theta)$ are orthonormal on the interval $0 \leqslant \theta \leqslant \pi$. Consider the integral

$$I_1 = \int_0^\pi d\theta \, \langle \psi_{\mu'}^L | \theta \rangle \langle \theta | \psi_{\mu'}^R \rangle$$
$$= \frac{1}{2\pi} \int_0^\pi \frac{d\theta}{\sin(\theta)} \exp\left[ -i\mu' \ln\left( \tan\frac{\theta}{2} \right) \right] \exp\left[ i\mu \ln\left( \tan\left( \frac{\theta}{2} \right) \right) \right] \quad \text{(A.1)}$$

Now make the change of variables

$$x = \ln\left( \tan\frac{\theta}{2} \right) \quad \text{(A.2)}$$

Then Eq. (A.1) becomes

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, e^{-i(\mu' - \mu)x} = \delta(\mu' - \mu) \quad \text{(A.3)}$$

### A.2. Completeness

We wish to check if $\int_{-\infty}^{\infty} d\mu \, \psi_\mu^R(\theta) \, \psi_\mu^{L*}(\theta') = \delta(\theta - \theta')$. Let us examine the integral

$$I_2 = \int_{-\infty}^{\infty} d\mu \int_0^\pi d\theta \, f(\theta) \, \psi_{\mu'}^R(\theta) \, \psi_\mu^{L*}(\theta')$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, g(x) \int_{-\infty}^{\infty} d\mu \, e^{i\mu(x - x')} \quad \text{(A.4)}$$

where $f(\theta) = g(x)$. But from the definition of the Dirac delta function,

$$I_2 = \int_{-\infty}^{\infty} dx \, g(x) \, \delta(x - x') = g(x') = f(\theta') \quad \text{(A.5)}$$

Thus, the eigenstates are complete!

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